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Kumaraswamy Transmuted Exponentiated Additive Weibull Distribution

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Abstract

This paper introduces a new lifetime model which is a generalization of the transmuted exponentiated additive Weibull distribution by using the Kumaraswamy generalized (Kw-G) distribution. With the particular case no less than **seventy nine** sub models as special cases, the so-called Kumaraswamy transmuted exponentiated additive Weibull distribution, introduced by Cordeiro and de Castro (2011) is one of this particular cases. Further, expressions for several probabilistic measures are provided, such as probability density function, hazard function, moments, quantile function, mean, variance and median, moment generation function, Rényi and q entropies, order estatistics, etc. Inference is maximum likelihood based and the usefulness of the model is showed by using a real dataset.

Keywords: Additive Weibull distribution, order statistics, maximum likelihood estimation, Rényi and q entropies, goodness of fit, moment generating function.

1. Introduction

Aiming improve the modeling of survival data, there has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models. As a result, significant progress has been made towards the generalization of some well-known lifetime models, which have been successfully applied to problems arising in several areas of research.

There are many distributions for modeling such data among the known parametric models, the most popular are the gamma, lognormal and the Weibull distributions, which is the most popular ones. However, with a limited hazard shapes, monotonic increase, decrease and constant, the Weibull distribution is not able to fit data sets with different hazard shapes as bathtub or upside down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering and biological studies. For many years, researchers have been developing various extensions and modified forms of the Weibull distribution, with number of parameters ranging from 2 to 7, see for example: Pham and Lai (2007) that present a review of some of the generalizations or modifications of Weibull distribution; the two-parameter flexible Weibull extension of Bebbington *et al.* (2007) that the hazard function can be increasing, decreasing or bathtub shaped; a three parameter model, called exponentiated Weibull distribution, introduced by Mudholkar and Srivastava (1993); another three-parameter one, introduced by Marshall and Olkin (1997) and called extended Weibull distribution; proposed by Xie *et al.* (2002); Xie and Lai (1995), a three parameter modified Weibull extension and a four parameter additive Weibull (AW) distribution, both with a bathtub shaped hazard function; the transmuted additive Weibull introduced by Elbatal and Aryal (2013) and Al-Babtain *et al.* (2015) introduced a new seven parameter model called the Kumaraswamy transmuted exponentiated modified Weibull distribution.

In this paper we introduce a new eight parameters model as a competitive extension for the Weibull distribution using the Kumaraswamy-generalized (Kw-G) distribution. The new model is very flexible in accommodating all forms of the hazard rate function by changing its parameter values, so it seems to be an important distribution that can be used. Another importance of the proposed model that it is very flexible model that approaches to different distributions when their parameters are changed. The new distribution is reffered to as the *Kumaraswamy transmuted exponentiated additive Weibull* (Kw-TEAW) distribution which extends all recent developments on the additive Weibull such as the transmuted exponentiated additive Weibull, Kumaraswamy transmuted exponentiated modified Weibull, transmuted modified Weibull introduced by Khan and King (2013), modified Weibull introduced by M. and Zaindin (2013) and additive Weibull introduced by Xie and Lai (1995) among others.

This paper is outlined as follows. In Section 2 we demonstrate the subject distribution and the mixture representation of its probability density function (pdf), cumulative distribution function (cdf), reliability function, hazard rate and cumulative hazard rate. The graphical presentation and sub-models of the Kw-TEAW are also provided in this section. The statistical properties include quantile functions, moments, moment generating functions, incomplete moments, mean deviations, moments of the residual life and moments of the reversed residual life are derived in Section 2.1. The order statistics and their moments are investigated in Section 4. In Section 5, We discuss maximum likelihood estimation of the model parameters. In Section 6, the Kw-TEAW distribution is applied to a real data sets to illustrate the potentiality of the new distribution for lifetime data modeling. Finally, we provide some concluding remarks in Section 7.

2. The Kw-TEAW Distribution

In this section, we present the Kw-TEAW distribution and its sub-models as follows:

Proposition 2.1. *Let X a positive random variable with Kw-TEAW distribution with vector parameters $\upsilon = (\alpha, \beta, \gamma, \theta, \delta, \lambda, a, b)$. The cumulative distribution function is defined as*

$$F(x) = 1 - \left\{1 - \left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{a\delta} \left[1 + \lambda - \lambda\left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b}, \tag{1}$$

where α and γ are scale parameter representing the characteristic life, θ, β, δ, a and b are the shape parameters representing the different patterns of the Kw-TEAW and λ is the transmuted parameter.

Proof: *The proof is imediately as follows: A new six parameter additive Weibull was introduced recently. Let X be a positive random variable with additive Weibull, it cumulative distribution function (cdf) is given by*

$$F(x) = \left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{\delta} \left[1 + \lambda - \lambda\left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{\delta}\right], \tag{2}$$

where $\alpha, \beta, \gamma, \theta \geq 0$ with $\theta < 1 < \beta$ (or $\beta < 1 < \theta$), θ and β are the shape parameters and α and γ are scale parameters.

The corresponding pdf of (2) is

$$\begin{aligned} f(x) &= \delta\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)} \left[1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right]^{\delta-1} \\ &\quad \times \left\{1 + \lambda - 2\lambda\left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{\delta}\right\}. \end{aligned} \tag{3}$$

Cordeiro and de Castro (2011) defined the Kw-G distribution by following general construction. For an arbitrary baseline cdf, $G(x)$, of a positive random variable X, the generalized class of distributions can be defined by

$$F(x) = 1 - \{1 - G(x)^{a}\}^{b}, \tag{4}$$

where $g(x)=dG(x)/dx$ and a and b are two extra positive shape parameters which govern skewness and tail weights. The Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(x) = abg(x)G(x)^{a-1}\{1 - G(x)^{a}\}^{b}. \tag{5}$$

Hence, each new Kw-G distribution can be generated from a specified G distribution.

Thus, as a proof, the equations (2) and (3) are inserted in equations (4) and (5), respectively, and we obtain the Kw-TEAW distribution.

□

The corresponding pdf of the Kw-TEAW is given by

$$\begin{aligned} f(x) &= ab\delta e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)} \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) \left[1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right]^{a\delta-1} \left\{1 - \left[1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right]^{a\delta} \left[1 + \lambda - \lambda\left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b-1} \\ &\quad \times \left\{1 + \lambda - 2\lambda\left[1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right]^{\delta}\right\} \left\{1 + \lambda - \lambda\left[1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right]^{\delta}\right\}^{a-1}. \end{aligned} \tag{6}$$

Furthermore, the reliability function $R(x)$, hazard rate function $h(x)$ and and cumulative hazard rate function $H(x)$ of the random variable X are given, respectively, by

$$R(x)=\left[\left\{1-\left(1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{a\delta}\left[1+\lambda-\lambda\left(1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b}\right],$$

$$\begin{aligned}h(x) &= ab\delta\left[\alpha\theta x^{\theta-1}e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}+\gamma\beta x^{\beta-1}e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]\left\{1-\left(1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{a\delta}\left[1+\lambda-\lambda\left(1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{-1}\\ &\left[1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]^{a\delta-1}\left\{1+\lambda-2\lambda\left[1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]^{\delta}\right\}\left\{1+\lambda-\lambda\left[1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]^{\delta}\right\}^{a-1},\end{aligned}$$

$$H(x)=-\ln\left[\left\{1-\left(1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{a\delta}\left[1+\lambda-\lambda\left(1-e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b}\right]. \tag{7}$$

2.1 Mixture Representation for cdf and pdf

Expansions for equations (1) and (6) can be derived using the series expansion

$$(1-z)^{k}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(k+1)}{j!\Gamma(k-j+1)}z^{j},\ |z|<1,\ k>0.$$

Then, the cdf of the Kw-TEAW in (1) can be expressed in the mixture form

$$F(x)=\sum_{j,i,l,w=0}^{\infty}s_{j,i,l,w}e^{-w\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}, \tag{8}$$

where

$$s_{j,i,l,w}=\frac{(-1)^{j+i+l+w}\Gamma(bj+1)\Gamma(ai+1)\Gamma(\delta ai+l+1)\lambda^{l}(1+\lambda)^{ai-l}}{j!i!l!w!\Gamma(2-j)\Gamma(bj+1-i)\Gamma(ai+1-l)\Gamma(\delta ai+l+1-w)}.$$

The pdf of the Kw-TEAW can be expressed in the mixture form

$$f(x)=\sum_{j,i,lw=0}^{\infty}\zeta_{j,i,l,w}\left[\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right]e^{-(w+1)\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}, \tag{9}$$

where

$$\zeta_{j,i,l,w}=ab\delta\frac{(-1)^{j+i+l+w}\Gamma(b)\Gamma(aj+a)\Gamma[\delta(aj+a+i+l)]2^{l}\lambda^{l+i}(1+\lambda)^{aj+a-i-l}}{j!i!l!w!\Gamma(b-j)\Gamma(aj+a-i)\Gamma(2+l)\Gamma[\delta(aj+a+i+l)-w]}. \tag{10}$$

Further, the Kw-TEAW density function can be expressed as a mixture of additive Weibull densities. Thus, some of its mathematical properties can be obtained directly from the properties of the additive Weibull distribution. Therefore equation (6) can be also expressed as

$$f(x)=\sum_{j,i,lw=0}^{\infty}\frac{\zeta_{j,i,l,w}}{w+1}g(x;\theta,\beta,\alpha(w+1),\gamma(w+1)), \tag{11}$$

where $g(x;\theta,\beta,\alpha(w+1),\gamma(w+1))$ denotes to the AW pdf i.e., $X\sim\text{AW}(\theta,\theta,\beta,\alpha(w+1),\gamma(w+1))$.

The Kw-TEAW model is very flexible model that approaches to different distributions. It includes as special cases seventy nine sub-models when its parameters vary as presented in Table 1. Simply by replacing the values of the parameters of Kw-TEAW as indicated in Table 1, is possible to write the particlar cases of this model, wheter these are new or known.

Figure B provides some plots of the Kw-TEAW density and hazard curves for different values of the parameters $a, b, \alpha, \beta, \gamma, \theta, \delta$ and λ.

3. Some Statistical Properties

The statistical properties of the Kw-TEAW distribution including quantile and random number generation, moments, moment generating function, incomplete moments, mean deviations and Rényi and q entropies are discussed in this section.

3.1 Quantile Function

The quantile function (qf) of X, where $X \sim$ Kw-TEAW$(\alpha, \beta, \gamma, \theta, \delta, \lambda)$, is obtained by inverting (2) as

$$\alpha x_q^{\theta} + \gamma x_q^{\beta} + \ln\left\{1 - \sqrt[\delta]{\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda m}}{2\lambda}}\right\} = 0, \tag{12}$$

where

$$m = \left[1 - (1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}}.$$

Since the above equation has no closed form solution in x_q, we have to use numerical methods to get the quantiles.

3.2 Moments

The rth moment, denoted by $\mu_r^{'}$, of the Kw-TEAW$(\alpha, \beta, \gamma, \theta, \delta, \lambda, a, b, x)$ is given by the following theorem.

Theorem 3.1. *If X is a continuous random variable has the Kw-TEAW$(\alpha, \beta, \gamma, \theta, \delta, \lambda, a, b, x)$, then the rth non-central moment of X, is given by*

$$\mu_r^{'} = \sum_{j,i,lw=0}^{\infty} \zeta_{j,i,l,w} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \gamma^k \Gamma\left(\frac{r+\theta+k\beta}{\theta}\right)}{k! \alpha^{\frac{r+k\beta}{\theta}} (w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}} + \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k \Gamma\left(\frac{r+\beta+k\theta}{\beta}\right)}{k! \gamma^{\frac{r+k\theta}{\beta}} (w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}} \right\}. \tag{13}$$

Proof: *By definition*

$$\mu_r^{'} = \int_0^{\infty} x^r f(x, \upsilon)\, dx = \sum_{j,i,lw=0}^{\infty} \zeta_{j,i,l,w} \int_0^{\infty} \left(\alpha\theta x^{r+\theta-1} + \gamma\beta x^{r+\beta-1}\right) e^{-(i+1)\left(\alpha x^{\theta} + \gamma x^{\beta}\right)} dx.$$

After some simplifications, we get

$$\mu_r^{'} = \sum_{j,i,l,w,k=0}^{\infty} \upsilon_{j,i,l,w,k} \left\{ \frac{\gamma^k \Gamma\left(\frac{r+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{r+k\beta}{\theta}} (w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}} + \frac{\alpha^k \Gamma\left(\frac{r+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{r+k\theta}{\beta}} (w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}} \right\},$$

where

$$\upsilon_{j,i,l,w,k} = \frac{(-1)^{j+i+l+w}}{j!i!l!w!k!} \frac{ab\delta\Gamma(b)\Gamma(aj+a)\Gamma[\delta(aj+a+i+l)] 2^l \lambda^{l+i} (1+\lambda)^{aj+a-i-l}}{\Gamma(b-j)\Gamma(aj+a-i)\Gamma(2+l)\Gamma[\delta(aj+a+i+l)-w]}.$$

□

The variation, skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships following.

Corollary 3.1. *Using the relation between the central moments and non-centeral moments, we can obtain the nth central moment, denoted by M_n, of a Kw-TEAW random variable as follows*

$$M_n = E(X-\mu)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} E(X^r),$$

where $E(X^r)$ is the on-central moments of the Kw-TEAW$(\alpha, \beta, \gamma, \theta, \delta, \lambda, a, b, x)$. Therefore the nth central moments of the Kw-TEAW$(\alpha, \beta, \gamma, \theta, \delta, \lambda, a, b, x)$, is given by

$$M_n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} \sum_{j,i,l,w,k=0}^{\infty} \upsilon_{j,i,l,w,k} \left\{ \frac{\gamma^k \Gamma\left(\frac{r+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{r+k\beta}{\theta}} (w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}} + \frac{\alpha^k \Gamma\left(\frac{r+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{r+k\theta}{\beta}} (w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}} \right\}.$$

3.3 Generating Function

The moment generating function of the Kw-TEAW is given by the following theorem

Theorem 3.2. *If X is a continuous random variable has the Kw-TEAW*(υ, x), *then the moment generating function of X, denoted by* $M_X(t) = E\left(e^{tX}\right)$, *is given as*

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{j,i,l,w,k=0}^{\infty}\upsilon_{j,i,l,w,k}\left\{\frac{\gamma^k\Gamma\left(\frac{r+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{r+k\beta}{\theta}}(w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}}+\frac{\alpha^k\Gamma\left(\frac{r+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{r+k\theta}{\beta}}(w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}}\right\}. \tag{14}$$

Proof: *By definition*

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} f(x,\upsilon)\,dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x,\upsilon)\,dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'. \end{aligned} \tag{15}$$

By substituting from equation (14) into (11), we obtain the moment generating function as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{j,i,l,w,k=0}^{\infty}\upsilon_{j,i,l,w,k}\left\{\frac{\gamma^k\Gamma\left(\frac{r+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{r+k\beta}{\theta}}(w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}}+\frac{\alpha^k\Gamma\left(\frac{r+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{r+k\theta}{\beta}}(w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}}\right\}.$$

□

3.4 Incomplete Moments

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The answers to many important questions in economics require more than justknowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well.

The *s-th* incomplete moments, denoted by $\varphi_s(t)$, of the Kw-TEAW random variable is given by

$$\varphi_s(t) = \int_0^t x^s f(x)\,dx,$$

Using equation (6) and the lower incomplete gamma function, we obtain

$$\varphi_s(t) = \sum_{j,i,l,w,k=0}^{\infty}\upsilon_{j,i,l,w,k}\left\{\frac{\gamma^k\Gamma\left(\frac{s+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{s+k\beta}{\theta}}(w+1)^{[k(\theta-\beta)-(\theta+s)]/\theta}}+\frac{\alpha^k\Gamma\left(\frac{s+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{s+k\theta}{\beta}}(w+1)^{[k(\beta-\theta)-(\beta+s)]/\beta}}\right\}. \tag{16}$$

The first incomplete moment of X, denoted by, $\varphi_1(t)$, is immediately calculated from equation (18) by setting $s = 1$ as

$$\varphi_1(t) = \sum_{j,i,l,w,k=0}^{\infty}\upsilon_{j,i,l,w,k}\left\{\frac{\gamma^k\Gamma\left(\frac{1+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{1+k\beta}{\theta}}(w+1)(w+1)^{[k(\theta-\beta)-(\theta+1)]/\theta}}+\frac{\alpha^k\Gamma\left(\frac{1+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{1+k\theta}{\beta}}(w+1)^{[k(\beta-\theta)-(\beta+1)]/\beta}}\right\}.$$

Another application of the first incomplete moment is related to the mean residual life and the mean waiting time (also known as mean inactivity time) given by

$$m_1(t;\theta) = (1-\varphi_1(t))\,R(t;\theta) - t$$

and

$$M_1(t;\theta) = t - (\varphi_1(t)\,F(t;\theta)),$$

respectively.

3.5 Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean $\delta_\mu(X) = E(|X - \mu_1^{'}|)$ and about the median $\left(\delta_\mu(X) = E(|X - M|)\right)$ of X can be, used as measures of spread in a population, expressed by

$$\delta_\mu(X) = \int_0^\infty \left|X - \mu_1^{'}\right| f(x)\,dx = 2\mu_1^{'} F\left(\mu_1^{'}\right) - 2\varphi_1\left(\mu_1^{'}\right), \tag{17}$$

and

$$\delta_M(X) = \int_0^\infty |X - M|\, f(x)\,dx = \mu_1^{'} - 2\varphi_1(M), \tag{18}$$

respectively, where $\mu_1^{'} = E(X)$ comes from (11), $F(\mu_1^{'})$ is simply calculated from (1), $\varphi_1(\mu_1^{'})$ is the first incomplete moments and M is the median of X.

The application of mean deviations refers to the Lorenz and Bonferroni curves defined by $L(p) = \varphi_1(q)/\mu_1^{'}$ and $B(p) = \varphi_1(q)/p\mu_1^{'}$, respectively, where $q = F^{-1}(p)$ can be computed fora given probability p by inverting (1) numerically. These curves are very useful in economics, reliability, demography, insurance and medicine.

3.6 Moments of the Residual Life

Several functions are defined related to the residual life. The failure rate function, mean residual life function and the left censored mean function, also called vitality function. It is well known that these three functions uniquely determine $F(x)$.

First, we present the nth moments of residual life, denoted by $m_n(t) = E((X - t)^n \mid X > t)$, $n = 1, 2, 3, ...,$ uniquely determine $F(x)$. In a general way, the nth moments of the residual life random variable is given by

$$m_n(t) = \frac{1}{1 - F(t)} \int_t^\infty (x - t)^n\, dF(x). \tag{19}$$

Another interesting function is the mean residual life function (MRL) or the life expectancy at age t, defined by $m_1(x) = E((X - x) \mid X > x)$, and it represents the expected additional life length for a unit which is alive at age x. Definitions 3.1 and Result 3.1 present, respectively, the $m_n(t)$ and $m_1(x)$.

Definition 3.1. *The nth moments of the residual life of X is given by*

$$m_n(t) \;=\; \frac{1}{R(t)} \sum_{r=0}^{n} \frac{(-1)^{n-r}\,\Gamma(n+1)\,t^{n-r}}{r!\,\Gamma(n-r+1)} \sum_{j,i,l,w,k=0}^{\infty} \upsilon_{j,i,l,w,k} \left\{ \frac{\gamma^k \Gamma\left(\frac{r+\theta+k\beta}{\theta}, \alpha(w+1)t^\theta\right)}{\alpha^{\frac{r+k\beta}{\theta}} (w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}} + \frac{\alpha^k \Gamma\left(\frac{r+\beta+k\theta}{\beta}, \gamma(w+1)t^\beta\right)}{\gamma^{\frac{r+k\theta}{\beta}} (w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}} \right\}.$$

Here we can use the upper incomplete gamma function defined by

$$\Gamma(a, b) = \int_b^\infty y^{a-1} e^{-y} dy.$$

Result 3.1. *The mean residual life is obtained by setting* $n = 1$ *in equation (22) and it is given by*

$$m_1(t) \;=\; \frac{1}{R(t)} \sum_{j,i,l,w,k=0}^{\infty} \upsilon_{j,i,l,w,k} \left\{ \frac{\gamma^k \Gamma\left(\frac{1+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{1+k\beta}{\theta}} (i+1)^{[k(\theta-\beta)-(\theta+1)]/\theta}} + \frac{\alpha^k \Gamma\left(\frac{1+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{1+k\theta}{\beta}} (i+1)^{[k(\beta-\theta)-(\beta+1)]/\beta}} \right\} - t.$$

Guess and Proschan (1988) gave an extensive coverage of possible applications of the mean residual life. The MRL has many applications in survival analysis in biomedical sciences, life insurance, maintenance and product quality control, economics and social studies, Demography and product tecnology (see Lai and Xie (2006)).

3.7 Moments of the Reversed Residual Life

Definition 3.2. *Let X be a random variable, usually representing the life length for a certain unit at age t (where this unit can have multiple interpretations). Then, the nth moment of the reversed residual life of X, is given by*

$$M_n(t) \;=\; \frac{1}{F(t)} \sum_{r=0}^{n} \frac{(-1)^{r}\,\Gamma(n+1)\,t^{n-r}}{r!\,\Gamma(n-r+1)} \sum_{j,i,l,w,k=0}^{\infty} \upsilon_{j,i,l,w,k} \left\{ \frac{\gamma^k \Gamma\left(\frac{r+\theta+k\beta}{\theta}, \alpha(w+1)t^\theta\right)}{\alpha^{\frac{r+k\beta}{\theta}} (w+1)^{[k(\theta-\beta)-(\theta+r)]/\theta}} + \frac{\alpha^k \Gamma\left(\frac{r+\beta+k\theta}{\beta}, \gamma(w+1)t^\beta\right)}{\gamma^{\frac{r+k\theta}{\beta}} (w+1)^{[k(\beta-\theta)-(\beta+r)]/\beta}} \right\}.$$

Here we can use the lower incomplete gamma function defined by $\Gamma(a, b) = \int_0^b y^{a-1} e^{-y} dy$.

Note that, the nth moment of the reversed residual life, denoted by

$$M_n(t) = E\left((t-X)^n \mid X \le t\right), \quad t > 0, \quad n = 1, 2, 3, ...,$$

uniquely determine $F(x)$. Then, the result presented in equation (20) is a result of the integral

$$M_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n \, dF(x). \tag{20}$$

Result 3.2. *The mean reversed residual life function, defined by* $M_1(t) = E\left((t-X) \mid X \le t\right)$, *is given by*

$$M_1(t) = t + \frac{1}{F(t)} \sum_{j,i,l,w,k=0}^{\infty} \upsilon_{j,i,l,w,k} \left\{ \frac{\gamma^k \Gamma\left(\frac{1+\theta+k\beta}{\theta}\right)}{\alpha^{\frac{s+k\beta}{\theta}} (w+1)^{\left[k(\theta-\beta)-(\theta+1)\right]/\theta}} + \frac{\alpha^k \Gamma\left(\frac{1+\beta+k\theta}{\beta}\right)}{\gamma^{\frac{s+k\theta}{\beta}} (w+1)^{\left[k(\beta-\theta)-(\beta+1)\right]/\beta}} \right\}.$$

Note that, the MRRL of the Kw-TEAW distribution can be obtained by setting $n = 1$ *in equation (20) presented in Definition 3.2*

The mean inactivity time (MIT) or mean waiting time (MWT) also called mean reversed residual life function, presented in 3.2, represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, x)$.

3.8 Rényi and q- Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty and is defined as

$$I_\kappa(X) = \frac{1}{1-\kappa} \log \int_{-\infty}^{\infty} f^\kappa(x)\, dx, \ \kappa > 0 \text{ and } \kappa \ne 1.$$

Hence, the Rényi entropy reduces to

$$I_\kappa(X) = \frac{1}{1-\kappa} \log \left\{ (ab\delta)^\kappa \sum_{L=0}^{\kappa} \binom{\kappa}{L} (\gamma\beta)^L (\alpha\theta)^{\kappa-L} \sum_{j.i.k.w.h=0}^{\infty} \Upsilon_{j.i.k.w.h} \Gamma\left(\frac{\beta(L+h) + \theta(\kappa-L) - \kappa + 1}{\theta} \right) \right\}, \tag{21}$$

where

$$\Upsilon_{j.i.k.w.h} = \frac{(-1)^{j+i+k+w+h}\, \Gamma[\kappa(\beta-1)+1]\, \Gamma[\kappa+1]}{j!i!k!w!h!\Gamma[\kappa(\beta-1)-j+1]\,\Gamma[\kappa-i+1]} \frac{\Gamma[a(\kappa+j)-\kappa+1]\,\Gamma[H]\,(1+\lambda)^{[a(\delta+j)-k-i]}\,\lambda^{k+i}2^i\alpha^{s+\kappa-L}(\kappa+w)^{O+h}}{\Gamma[a(\kappa+j)-\kappa-k+1]\,\Gamma[H-w]},$$

$$s = \left(\frac{L\theta - h\beta + \kappa - 1 - L\beta - \kappa\theta}{\theta} \right),$$

and

$$H = \delta a(\kappa+j) + \delta(i+k) + 1 - \kappa.$$

The q-entropy, say $H_q(x)$, is defined by

$$H_q(x) = \frac{1}{q-1} \log\left(1 - \int_{-\infty}^{\infty} f^q(x)\, dx\right), \ q > 0 \text{ and } q \ne 1.$$

From equation (20), we can easily obtain

$$H_q(x) = \frac{1}{q-1} \log\left(1 - \left\{ (ab\delta)^q \sum_{L=0}^{q} \binom{q}{L} (\gamma\beta)^L (\alpha\theta)^{q-L} \sum_{j.i.k.w.h=0}^{\infty} \tau_{j.i.k.w.h} \Gamma\left(\frac{\beta(L+h) + \theta(q-L) - q + 1}{\theta} \right) \right\}\right), \tag{22}$$

where

$$\tau_{j.i.k.w.h} = \frac{(-1)^{j+i+k+w+h}\, \Gamma[q(\beta-1)+1]\, \Gamma[q+1]}{j!i!k!w!h!\Gamma[q(\beta-1)-j+1]\,\Gamma[q-i+1]} \frac{\Gamma[a(q+j)-q+1]\,\Gamma[d]\,(1+\lambda)^{[a(\delta+j)-k-i]}\,\lambda^{k+i}2^i\alpha^{\pi+q-L}(q+w)^{\pi+h}}{\Gamma[a(q+j)-q-k+1]\,\Gamma[d-w]},$$

$$\pi = \left(\frac{L\theta - h\beta + q - 1 - L\beta - q\theta}{\theta} \right),$$

and

$$d = \delta a(q+j) + \delta(i+k) + 1 - q.$$

4. Order Statistics

The order statistics and their moments have great importance in many statistical problems and they have many applications in reliability analysis and life testing. The order statistics arise in the study of reliability of a system. The order statistics can represent the lifetimes of units or components of a reliability system. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the Kw-TEAW with cumulative distribution function, and the corresponding probability density function, as in (1) and (6), respectively. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the corresponding order statistics. Then the pdf of jth order statistics, say $Y = X_{(j:n)},\ 1 \leq j \leq n,$ denoted by $f_Y(x)$, is given by

$$\begin{aligned} f_Y(x) &= ab\delta\binom{n}{i}\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right)e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\left[1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]^{\delta a-1} \\ &\left\{1 + \lambda - \lambda\left[1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]^{\delta}\right\}^{a-1}\left\{1 + \lambda - 2\lambda\left[1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right]^{\delta}\right\} \\ &\left\{1 - \left\{1 - \left(1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{a\delta}\left[1 + \lambda - \lambda\left(1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b}\right\}^{i-1} \\ &\left[\left\{1 - \left(1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{a\delta}\left[1 + \lambda - \lambda\left(1 - e^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b}\right]^{b(n-i+1)-1}. \end{aligned} \tag{23}$$

Then, the pdf of Y can be expressed in a mixture form as

$$f_{i:n}(x) = \sum_{j,l,w,m,h=0}^{\infty} \varsigma_{j,l,w,m,h}\, g\left(x; \theta, \beta, \alpha(h+1), \gamma(h+1)\right), \tag{24}$$

where

$$\varsigma_{j,l,w,m,h} = ab\delta\binom{n}{i}\frac{(-1)^{j+l+w+m+h}(2)^{w}(\lambda)^{w+m}}{(h+1)(1+\lambda)^{-a(l+1)+w+m}}\binom{i-1}{j}\binom{al+1}{w}\binom{b(j+n-i+1)-1}{l}\binom{a-1}{m}\binom{\delta[a(l+1)+w+m]-1}{h},$$

and $g(x; \theta, \beta, \alpha(h+1), \gamma(h+1))$ is the additive Weibull density function with parameters $\theta, \beta, \alpha(h+1), \gamma(h+1)$, a and b. So, the density function of the Kw-TEAW order statistics is a mixture of AW densities. Based on equation (24), we can obtain some structural properties of Y from those AW properties.

The qth moment of the jth order statistics, $Y = X_{(j:n)}$, is given by

$$E\left(X^{q}_{(j:n)}\right) = \sum_{j,l,w,m,h=0}^{\infty} \varsigma_{(j,k,w,h)} E\left(Y^{q}_{\theta,\beta,\alpha(h+1),\gamma(h+1),a,b}\right), \tag{25}$$

where $Y_{\theta,\beta,\alpha(h+1),\gamma(h+1)} \sim AW(\theta, \beta, \alpha(h+1), \gamma(h+1))$.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in equation (25), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable AW distributions. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum\nolimits_{d=0}^{r-1}(-1)^{d}\binom{r-1}{d}E\left(X_{r-d:d}\right),\ r \geq 1.$$

The first four L-moments are given by: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. One simply can obtain the λ's for X from equation (25) with $q = 1$.

5. Parameter Estimation

This section provides a system of equations that can be utilized to determine the maximum likelihood estimates of the parameters of the Kw-TEAW distribution. Let $\mathbf{X} = (X_1, ..., X_n)$ be a random sample of the Kw-TEAW distribution with unknown parameter vector $\upsilon = (\alpha, \beta, \gamma, \delta, \theta, \lambda, a, b)^{T}$.

Then, the log-likelihood function $\ell(\upsilon)$, is given by

$$\begin{aligned}\ell(\upsilon) &= n\ln a + n\ln b + n\ln\delta + (a\delta-1)\sum_{i=1}^{n}\ln\left(1-e^{S_i}\right) + \sum_{i=1}^{n}\ln Z_i \\ &\quad -\sum_{i=1}^{n} S_i + \sum_{i=1}^{n}\ln K_i + (a-1)\sum_{i=1}^{n}\ln\{Q_i\}(b-1)\sum_{i=1}^{n}\ln\left\{1-\left[1-e^{S_i}\right]^{a\delta}\{Q_i\}^a\right\}. \end{aligned} \tag{26}$$

Therefore the score vector is

$$\mathbf{U}(\upsilon) = \frac{\partial\ell}{\partial\upsilon} = \left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\gamma},\frac{\partial\ell}{\partial\delta},\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b}\right)^T.$$

Let $Z_i = \alpha\theta x_i^{\theta-1} + \gamma\beta x_i^{\beta-1}$, $Q_i = 1+\lambda-\lambda\left[1-e^{-S_i}\right]^{\delta}$, $K_i = 1+\lambda-2\lambda\left[1-e^{-S_i}\right]^{\delta}$ and $S_i = \alpha x_i^{\theta} + \gamma x_i^{\beta}$,

$$\begin{aligned}\frac{\partial\ell(\upsilon)}{\partial\alpha} &= (a\delta-1)\sum_{i=1}^{n}\frac{e^{-S_i}x_i^{\theta}}{1-e^{-S_i}} + \sum_{i=1}^{n}\frac{\theta x_i^{\theta-1}}{Z_i} - 2\lambda\delta\sum_{i=1}^{n}\frac{x_i^{\theta}e^{S_i}\left(1-e^{-S_i}\right)^{\delta-1}}{K_i} - (a-1)\sum_{i=1}^{n}\frac{\lambda\delta e^{-S_i}x_i^{\theta}\left(1-e^{-S_i}\right)^{\delta-1}}{Q_i} \\ &\quad -\sum_{i=1}^{n} x_i^{\theta} + (a-1)\sum_{i=1}^{n}\frac{a\delta e^{S_i}x_i^{\theta}\left(1-e^{-S_i}\right)^{a\delta-1}[Q_i]^{a-1}\left[2\lambda\left(1-e^{-S_i}\right)^{\delta}-(1+\lambda)\right]}{1-(1-e^{-S_i})^{a\delta}[Q_i]^a}, \end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial\ell(\upsilon)}{\partial\beta} &= \gamma(a\delta-1)\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-S_i}\ln(x_i)}{1-e^{-S_i}} + \gamma\sum_{i=1}^{n}\frac{x_i^{\beta-1}(\beta\ln x_i+1)}{Z_i} - \gamma\delta\lambda(a-1)\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-S_i}\ln\left[x_i\left(1-e^{-S_i}\right)^{\delta-1}\right]}{Q_i} \\ &\quad + a\gamma\delta(b-1)\sum_{i=1}^{n}\frac{x_i^{\theta}e^{-S_i}(Q_i)^{a-1}\left[2\lambda\left(1-e^{-S_i}\right)^{\delta}-(1+\lambda)\right]}{1-(1-e^{-S_i})^{a\delta}(Q_i)^a}\ln\left[x_i\left(1-e^{-S_i}\right)^{a\delta-1}\right] \\ &\quad -\gamma\sum_{i=1}^{n}x_i^{\beta}\ln(x_i) - 2\gamma\delta\lambda\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-S_i}\ln\left[x_i\left(1-e^{-S_i}\right)^{\delta-1}\right]}{K_i}, \end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial\ell(\upsilon)}{\partial\gamma} &= (a\delta-1)\sum_{i=1}^{n}\frac{e^{-S_i}x_i^{\beta}}{1-e^{-S_i}} + \sum_{i=1}^{n}\frac{\beta x_i^{\beta-1}}{Z_i} - \sum_{i=1}^{n}x_i^{\beta} - \delta\lambda(a-1)\sum_{i=1}^{n}\frac{e^{-S_i}x_i^{\theta}\left(1-e^{-S_i}\right)^{\delta-1}}{Q_i} \\ &\quad + a\delta(b-1)\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-S_i}(Q_i)^{a-1}\left(1-e^{-S_i}\right)^{a\delta-1}\left[2\lambda\left(1-e^{-S_i}\right)^{\delta}-(1+\lambda)\right]}{1-(1-e^{-S_i})^{a\delta}(Q_i)^a} + 2\delta\lambda\sum_{i=1}^{n}\frac{x_i^{\theta}e^{-S_i}\left(1-e^{-S_i}\right)^{\delta-1}}{K_i}, \end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial\ell(\upsilon)}{\partial\delta} &= \frac{n}{\delta} + a\sum_{i=1}^{n}\ln\left(1-e^{-S_i}\right) - 2\lambda\sum_{i=1}^{n}\frac{\left(1-e^{-S_i}\right)^{\delta}\ln\left(1-e^{-S_i}\right)}{K_i} - \lambda(a-1)\sum_{i=1}^{n}\frac{\left(1-e^{-S_i}\right)^{\delta}\ln\left(1-e^{-S_i}\right)}{Q_i} \\ &\quad + a(b-1)\sum_{i=1}^{n}\frac{\left(1-e^{-S_i}\right)^{a\delta}(Q_i)^{a-1}\left[2\lambda\left(1-e^{-S_i}\right)^{\delta}-(1+\lambda)\right]\ln\left(1-e^{-S_i}\right)}{1-(1-e^{-S_i})^{a\delta}(Q_i)^a}, \end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial\ell(\upsilon)}{\partial\theta} &= (a\delta-1)\sum_{i=1}^{n}\frac{\alpha e^{-S_i}x_i^{\theta}}{1-e^{-S_i}} - \alpha\lambda\delta(a-1)\sum_{i=1}^{n}\frac{x_i^{\theta}e^{S_i}\ln\left[x_i\left(1-e^{-S_i}\right)^{\delta-1}\right]}{K_i} + \alpha\sum_{i=1}^{n}\frac{x_i^{\theta-1}(\theta\ln x_i+1)}{Z_i} \\ &\quad -\alpha\sum_{i=1}^{n}x_i^{\theta}\ln x_i - 2\alpha\lambda\delta\sum_{i=1}^{n}\frac{x_i^{\theta}e^{S_i}\ln\left[x_i\left(1-e^{-S_i}\right)^{\delta-1}\right]}{K_i} \\ &\quad + a\alpha\delta(b-1)\sum_{i=1}^{n}\frac{e^{S_i}\left[1+\lambda-\lambda\left(1-e^{-S_i}\right)^{\delta}\right]^{a-1}\left[2\lambda\left(1-e^{-S_i}\right)^{\delta}-(1+\lambda)\right]}{\left\{\ln\left[x_i(1-e^{-S_i})^{a\delta-1}\right]\right\}^{-1}\left\{1-(1-e^{-S_i})^{a\delta}\left[1+\lambda-\lambda(1-e^{-S_i})^{\delta}\right]^a\right\}}, \end{aligned} \tag{31}$$

$$\frac{\partial \ell(\upsilon)}{\partial \lambda} = (a-1)\sum_{i=1}^{n} \frac{1-\left(1-e^{-S_i}\right)^{\delta}}{1+\lambda-\lambda\left(1-e^{-S_i}\right)^{\delta}} + a(b-1)\sum_{i=1}^{n} \frac{(Q_i)^{a-1}\left(1-e^{-S_i}\right)^{a\delta}\left[1-\left(1-e^{-S_i}\right)\right]}{(Q_i)^{a}\left(1-e^{-S_i}\right)^{a\delta}} + \sum_{i=1}^{n} \frac{1-2\left(1-e^{-S_i}\right)^{\delta}}{K_i}, \quad (32)$$

$$\frac{\partial \ell(\upsilon)}{\partial a} = \frac{n}{a} + \delta\sum_{i=1}^{n} \ln\left(1-e^{-S_i}\right) + \sum_{i=1}^{n} \ln(Q_i) - \delta(b-1)\sum_{i=1}^{n} \frac{Q_i\left(1-e^{-S_i}\right)^{\delta}\left[\ln\left(1-e^{-S_i}\right)Q_i\right]}{1-\left(1-e^{-S_i}\right)^{a\delta}(Q_i)^{a}}, \quad (33)$$

and

$$\frac{\partial \ell(\upsilon)}{\partial b} = \frac{n}{b}\sum_{i=1}^{n} \ln\left[1-\left(1-e^{-S_i}\right)^{a\delta} Q_i\right]. \quad (34)$$

We can find the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\upsilon}) = 0$, and solving them simultaneously yields the ML estimators $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \widehat{\theta}, \widehat{\lambda}, \widehat{a}$ and $\widehat{b}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm.

In order to compute the standard error and asymptotic confidence interval we use the usual large sample approximation (Migon *et al.*, 2014) in which the maximum likelihood estimators of $\boldsymbol{\upsilon}$ can be treated as being approximately hepta-variate normal. For example, as $n \to \infty$ the asymptotic distribution of the MLE $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \widehat{\theta}, \widehat{\lambda}, \widehat{a}, \widehat{b})$, is given by,

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\gamma} \\ \widehat{\delta} \\ \widehat{\theta} \\ \widehat{\lambda} \\ \widehat{a} \\ \widehat{b} \end{pmatrix} \sim N\left[\begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\gamma} \\ \widehat{\delta} \\ \widehat{\theta} \\ \widehat{\lambda} \\ \widehat{a} \\ \widehat{b} \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \dots & \hat{V}_{18} \\ \vdots & \ddots & \vdots \\ \hat{V}_{81} & \dots & \hat{V}_{88} \end{pmatrix}\right], \quad (35)$$

with, $\hat{V}_{ij} = V_{ij}\mid_{\theta=\hat{\theta}}$ and it is determined by the inverse of Fisher information that can be easily obtained since the second order derivatives of the log-likelihood function exist for all the eigth parameters of Kw-TEAW distribution. Thus, the approximate $100(1-\phi)\%$ confidence intervals for $\alpha, \beta, \gamma, \delta, \theta, \lambda, a$ and b can be determined as:

$$\widehat{\alpha} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{11}}, \qquad \widehat{\beta} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{22}}, \qquad \widehat{\gamma} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{33}}, \qquad \widehat{\delta} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{44}},$$

$$\widehat{\theta} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{55}}, \qquad \widehat{\lambda} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{66}}, \qquad \widehat{a} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{77}}, \qquad \widehat{b} \pm Z_{\frac{\phi}{2}}\sqrt{\widehat{V}_{88}},$$

where $Z_{\frac{\phi}{2}}$ is the upper ϕth percentile of the standard normal distribution. The applicability of the estimation method and the asymptotic confidence intervals are as follows, in Section 6.

6. Application

In this section, we provide an application of the Kw-TEAW distribution to show the importance and usefulness of the new model. For that, we use the data works with nicotine measurements, made from several brands of cigarettes in 1998, collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection.

The report entitled tar, nicotine, and carbon monoxide of the smoke of 1206 varieties of domestic cigarettes for the year of 1998 consists of the data sets and some information about the source of the data, smokers behavior and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. The free form data set can be found at *http://pw1.netcom.com/rdavis2/smoke.html*. We analyzed data on nicotine, measured in milligrams per cigarette, from several cigarette brands and the TTT Plot of the rimes can be seen in Figure **??**-(a).

Table 2 shows the numerical values of the MLEs, the estimated standard error and four different selection criterias: -2 log, AIC, AICC and BIC. The adjustment of the model Kw-TEAW can be seen in Figure **??**-(b). Observe that all model are nested and the lowest value of -2 log likelihood observed was for the Kw-TEAW model, as expected.

6.1 Global and Local Influence to Kw-TEAW Estimated Model

In this section we will make an analysis of global and local influence for the data set given, using the Kw-TEAW model.

The first tool to assess the sensitivity analysis are measures of global influence. Starting with the case-deletion, that we study the effect of withdrawal of the *i*th element sampled. The first measure of global influence analysis is known as generalized Cook's distance, which is defined as the standard norm of $\boldsymbol{\psi}_i = (\alpha_i, \beta_i, \gamma_i, \delta_i, \theta_i, \lambda_i, a_i, b_i)$ and $\hat{\boldsymbol{\psi}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\theta}, \hat{\lambda}, \hat{a}, \hat{b})$ and is given by

$$CD_i(\boldsymbol{\psi}) = \left[\ \boldsymbol{\psi}_i - \hat{\boldsymbol{\psi}}\ \right]^T \left[-\ddot{L}(\boldsymbol{\psi})\right]\left[\ \boldsymbol{\psi}_i - \hat{\boldsymbol{\psi}}\ \right], \tag{36}$$

where $\ddot{L}(\boldsymbol{\psi})$ can be approximated by the estimated covariance and variance matrix.

Another way to measure the global influence is through the difference in likelihoods given by

$$LD_i(\boldsymbol{\psi}) = 2\left\{l(\hat{\boldsymbol{\psi}}) - l(\boldsymbol{\psi}_i)\right\}. \tag{37}$$

Figures 5-a and b show us, respectively, the Cook's generalized and likelihood distances where we could see possible influence points.

Furthermore, we know that the main objective of the local influence method is to evaluate changes in the results from the analysis when small perturbations are incorporated in the model and/or in the data. If such perturbations provoke disproportionate effects, it can be an indication that the model is fitted inadequately or serious departures from the assumptions of the model may exist.

In order to analyse the local influence, here we consider the response variable perturbation, ie, we will consider that each t_i is peturbed as $t_{im} = t_i + m_i V_t$, where V_t is a scale factor that may be the estimated standard deviation of T and $m_i \in \mathbb{R}$. Then, the perturbed log-likelihood function becomes expressed as

$$\begin{aligned}\ell\left(\psi|\mathbf{t},\mathbf{m}\right) \quad = \quad & n\ln a + n\ln b + n\ln\delta + (a\delta - 1)\sum\nolimits_{i=1}^{n}\ln\left(1 - e^{S_{im}}\right) + \sum\nolimits_{i=1}^{n}\ln Z_{im} \\ & - \sum\nolimits_{i=1}^{n} S_{im} + \sum\nolimits_{i=1}^{n}\ln K_{im} + (a-1)\sum\nolimits_{i=1}^{n}\ln\left\{Q_{im}\right\} \\ & .\,(b-1)\sum\nolimits_{i=1}^{n}\ln\left\{1 - \left[1 - e^{S_{im}}\right]^{a\delta}\left\{Q_{im}\right\}^{a}\right\},\end{aligned} \tag{38}$$

where $Z_{im} = \alpha\theta t_{im}^{\theta-1} + \gamma\beta t_{im}^{\beta-1}$, $Q_{im} = 1 + \lambda - \lambda\left[1 - e^{-S_{im}}\right]^{\delta}$, $K_{im} = 1 + \lambda - 2\lambda\left[1 - e^{-S_{im}}\right]^{\delta}$ and $S_{im} = \alpha t_{im}^{\theta} + \gamma t_{im}^{\beta}$.

Figures 5-a and b, show us, respectively, the the Cook's generalized and likelihood distances and it is possible to see that the pertubation provoke some disproportionate effects.

After analyse the Figures 4 and 5, we can see the distinction of two observations in relation to others. Furthermore, we made a residual analyse by using the Martingale-type and deviance, see for example McCullagh and Nelder (1989), Barlow and Prentice (1988) and Therneau *et al.* (1990).

The first one, martingale-type residual, was introduced by Therneau *et al.* (1990) and was firstly used in a counting processes and that ones are skewed and have a maximum value at +1 and a minimum value at $-\infty$. By considering the Kw-TEAW model, the martingale-type residual can be written as

$$r_{M_i} = 1 + \ln\left[\left\{1 - \left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{a\delta}\left[1 + \lambda - \lambda\left(1 - e^{-\left(\alpha x^{\theta} + \gamma x^{\beta}\right)}\right)^{\delta}\right]^{a}\right\}^{b}\right], \tag{39}$$

where $i = 1, \ldots, n$.

In addition, it is possible to use the deviance residual that has been widely applied in GLMs (generalized linear models). This one was proposed by the same authors (Therneau *et al.* (1990)) and it is a transformation of the martingale residual to attenuate the skewness. In our case, the deviance residuals is given by

$$r_{D_i} = \text{sign}(\hat{r}_{M_i})\left[-2\left(\hat{r}_{M_i} + \log\left(1 - \hat{r}_{M_i}\right)\right)\right], \tag{40}$$

$i = 1, \ldots, n$. Figures 6 show, respectively, the Martingale and deviance residuals.

In our case is clear that the observation $I = 30, 127$ can be influential points. In order to reveal the impact of these, the relative changes was measured as

$$\mathrm{RC}_{\zeta_j} = \left| \frac{\hat{\zeta}_j - \hat{\zeta}_{j(I)}}{\hat{\zeta}_j} \right| \times 100\%, \qquad j = 1, \ldots, p+1,$$

where $\hat{\zeta}_{j(I)}$ denotes the MLE of ζ_j after the set I of observations has been removed. Suggested by Lee *et al.* (2006), we use the total and maximum relative changes and the likelihood displacement given by

To reveal the impact of the detected influential observations, we use three measures defined by Lee *et al.* (2006),

$$\mathrm{TRC} = \sum_{i=1}^{n_p} \left| \frac{\hat{\zeta}_i - \hat{\zeta}_i^0}{\hat{\zeta}_i} \right|, \quad \mathrm{MRC} = \max_i \left| \frac{\hat{\zeta}_i - \hat{\zeta}_i^0}{\hat{\zeta}_i} \right| \quad \text{and} \quad LD_{(I)}(\boldsymbol{\zeta}) = 2\{l(\hat{\boldsymbol{\zeta}} - l(\hat{\boldsymbol{\zeta}}^0)\},$$

where TRC is the total relative changes, MRC the maximum relative changes and LD the likelihood displacement, with n_p (the number of parameters) and $\hat{\boldsymbol{\zeta}}^0$ denotes MLE of $\boldsymbol{\zeta}$ after the set I of observations has been removed. Table 3 show us the impact of these two observations.

Note that, when we withdrew the 10 most influential points, β parameter was the most affected. Then, the Kw-TEAW was re-fitted to the data, Table 4 shows the MLEs and Figure 7 shows the empirical and re-adjusted curves.

7. Conclusions

In this paper, we propose a new model, called the Kumaraswamy transmuted exponentiated additive Weibull (Kw-TEAW) distribution, which extends the transmuted exponentiated additive Weibull (TEAW) distribution and some other well known distributions in the literature. An obvious reason for generalizing a standard distribution is the fact that the generalization provides more flexibility to analyze real life data.

In fact, the Kw-TEAW distribution is motivated by the wide use of the Weibull distribution in practice and also its hazard rate function very flexible in accommodating all forms of the hazard. Some of its mathematical and statistical properties was presented beside the explicit expressions for the ordinary and generating function, moments residual life, moments of the reversed residual life and Rényi and q entropies. Finally, we illustrate the usefulness of the model showing that this one provides consistently better fit than the other nested models mentioned above. We hope that the proposed model will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

There are a large number of possible extensions of the current work. The presence of covariates, as well as of long-term survivals, is very common in practice. Our approach should be investigate in both contexts. A possible approach is to consider the regression schemes adopted by Achcar and Louzada-Neto (1992) and Perdona and Louzada (2011), respectively. Other generalisation can be obtained as in Flores *et al.* (2013), which proposes a complementary exponential power series distribution, which arises on latent complementary risks scenarios, where the lifetime associated with a particular risk is not observable, rather we observe only the maximum lifetime value among all hazards.

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Appendix

A. Tables

Table 1. Sub-models of the Kw-TEAW $(\alpha, \beta, \gamma, \theta, \delta, \lambda, a, b)$.

No.	Distribution	α	β	γ	θ	δ	λ	a	b	Author
1	kw-EAW	α	β	γ	θ	δ	0	a	b	–
2	kw-TAW	α	β	γ	θ	1	λ	a	b	–
3	Kw-AW	α	β	γ	θ	1	0	a	b	–
4	Kw-TEME	α	1	γ	1	δ	λ	a	b	–
5	Kw-TME	α	1	γ	1	1	λ	a	b	–
6	kw-EME	α	1	γ	1	δ	0	a	b	–
7	kw-ME	α	1	γ	1	1	0	a	b	–
8	GTEAW	α	β	γ	θ	δ	λ	a	1	–
9	GEAW	α	β	γ	θ	δ	0	a	1	–
10	GTAW	α	β	γ	θ	1	λ	a	1	–
11	GAW	α	β	γ	θ	1	0	a	1	–
12	GTEME	α	1	γ	1	δ	λ	a	1	–
13	GTME	α	1	γ	1	1	λ	a	1	–
14	GEME	α	1	γ	1	δ	0	a	1	–
15	GME	α	1	γ	1	1	0	a	1	–
16	TEAW	α	β	γ	θ	δ	λ	1	1	–
17	TAW	α	β	γ	θ	1	λ	1	1	Elbatal and Aryal (2013)
18	EAW	α	β	γ	θ	δ	0	1	1	–
19	AW	α	β	γ	θ	1	0	1	1	Xie and Lai (1995)
20	TEME	α	1	γ	1	δ	λ	1	1	–
21	TME	α	1	γ	1	1	λ	1	1	Elbatal and Aryal (2013)
22	EME	α	1	γ	1	δ	0	1	1	–
23	ME	α	1	γ	1	1	0	1	1	Elbatal and Aryal (2013)
24	Kw-TEMW	α	β	γ	1	δ	λ	a	b	Al-Babtain *et al.* (2015)
25	ETR	α	2	γ	1	δ	0	a	1	–
26	New-ER	0	2	γ	1	δ	0	a	1	–
27	New-EE	α	β	0	1	δ	0	a	1	–
28	Kw-TELFR	α	2	γ	1	δ	λ	a	b	–
29	Kw-TEW	0	β	γ	1	δ	λ	a	b	–
30	Kw-TER	0	2	γ	1	δ	λ	a	b	–
31	Kw-TEE	α	β	0	1	δ	λ	a	b	–
32	Kw-TMW	α	β	γ	1	1	λ	a	b	–
33	Kw-TLFR	α	2	γ	1	1	λ	a	b	–
34	Kw-TW	0	β	γ	1	1	λ	a	b	–
35	Kw-TR	0	2	γ	1	1	λ	a	b	–
36	Kw-TE	α	β	0	1	δ	λ	a	b	–
37	Kw-EMW	α	β	γ	1	δ	0	a	b	–
38	Kw-MW	α	β	γ	1	1	0	a	b	Cordeiro *et al.* (2014)
39	Kw-EW	0	β	γ	1	δ	0	a	b	–
40	ETEMW	α	β	γ	1	δ	λ	a	1	–
41	ETELFR	α	β	γ	1	δ	1	a	2	–
42	ETEW	0	β	γ	1	δ	λ	a	1	–
43	ETGR	0	2	γ	1	δ	λ	a	1	Afify *et al.* (2015)
44	ETEE	α	β	0	1	δ	λ	a	1	–
45	ETMW	α	β	γ	1	1	λ	a	1	–
46	ETLFR	α	2	γ	1	1	λ	a	1	–
47	ETW	0	β	γ	1	1	λ	a	1	–
48	ETR	0	2	γ	1	1	λ	a	1	–
49	ETE	α	β	0	1	1	λ	a	1	–
50	New-EMW	α	β	γ	1	δ	0	a	1	–

Continuing...

No.	Distribution	α	β	γ	θ	δ	λ	a	b	Author
51	New-ELFR	α	2	γ	1	δ	0	a	1	–
52	New-EW	0	β	γ	1	δ	0	a	1	–
53	Kw-EE	0	β	0	1	δ	λ	a	b	Gomes *et al.* (2014)
54	Kw-LFR	α	2	γ	1	1	0	a	b	Al-Babtain *et al.* (2015)
55	Kw-ELFR	α	2	γ	1	δ	0	a	b	Elbatal (2011)
56	Kw-ER	0	2	γ	1	δ	0	a	b	Gomes *et al*. (2014)
57	EMW	α	β	γ	1	1	0	a	1	Elbatal (2011)
58	EW	0	β	γ	1	δ	0	a	1	S. and K. (1993)
59	Kw-W	0	β	γ	1	1	0	a	b	Cordeiro and de Castro (2011)
60	Kw-R	0	2	γ	1	1	0	a	b	–
61	Kw-E	α	β	0	1	1	0	a	b	–
62	EE	α	β	0	1	1	λ	a	1	Gupta *et al.* (1998)
63	TEMW	α	β	γ	1	δ	λ	1	1	Eltehiwy and Ashour (2013)
64	TELFR	α	2	γ	1	δ	λ	1	1	–
65	TEW	0	β	γ	1	δ	λ	1	1	–
66	TER	0	2	0	1	δ	λ	1	1	Merovci (2013a)
67	TEE	α	β	γ	1	δ	λ	1	1	Merovci (2013b)
68	TMW	α	β	γ	1	1	λ	1	1	Khan and King (2013)
69	TLFR	α	2	γ	1	1	λ	1	1	–
70	TW	0	β	γ	1	1	λ	1	1	Aryal and Tsokos (2011)
71	TR	0	2	γ	1	δ	λ	1	1	Khan and King (2013)
72	TE	α	β	0	1	1	λ	1	1	Shaw and Buckley (2007)
73	ELFR	α	2	γ	1	δ	0	1	1	M. and Zaindin (2013)
74	ER	0	2	γ	1	δ	0	1	1	Kundu and Raqab (2005)
75	MW	α	β	γ	1	1	0	1	1	M. and Zaindin (2013)
76	LFR	α	2	γ	1	1	0	1	1	–
77	W	0	β	γ	1	1	0	1	1	Weibull (1951)
78	R	0	2	γ	1	1	0	1	1	Rayleigh (1880)
79	E	0	1	γ	1	1	0	1	1	–

Table 2. Estimates of parameters and confidence interval of Kw-TEAW model and some nested ones.

Model	Parameters	Estimate	Standard Error	Selection Criteria $-2\log$	AIC	AICC	BIC
Kw-TEAW	α	1.027	1.637	216.0	232.0	232.4	262.7
	β	2.033	1.989				
	λ	−0.606	0.513				
	θ	0.369	0.355				
	γ	1.218	2.470				
	a	1.385	0.944				
	b	1.649	7.173				
	δ	2.896	4.736				
TAW	α	0.388	0.354	218.1	228.1	228.3	247.4
	β	2.664	0.314				
	λ	−0.708	0.213				
	θ	1.216	0.517				
	γ	1.172	0.347				
AW	α	0.426	0.167	217.6	225.6	225.8	241.0
	β	2.652	0.235				
	θ	0.700	0.221				
	γ	1.245	0.187				
TEMW	α	0.722	0.501	217.1	227.1	227.2	246.3
	β	2.599	0.272				
	λ	−0.629	0.230				
	γ	1.177	0.265				
	δ	1.525	0.494				
EMW	α	0.453	0.305	218.9	226.9	227.0	242.3
	β	2.841	0.216				
	γ	1.068	0.133				
	δ	1.626	0.424				
EW	β	3.063	0.354	226.3	232.3	232.4	243.9
	γ	0.947	0.173				
	δ	0.812	0.152				
W	β	2.719	0.114	227.6	231.6	231.6	239.2
	γ	1.047	0.022				

Table 3. Some influence measures of set I.

Parameters	RC/100	TRC	MRC	LD
μ	0.484	11.370	6.012	12.1
β	6.012			
λ	1.635			
θ	0.856			
σ	0.367			
a	0.458			
b	1.454			
δ	0.103			

Table 4. MLEs considering the Kw-TEAW model after been removed the set I.

Parameters	Estimate	Standard Error	Confidence Interval (95%)	
			Lower	Upper
μ	1.9903	1.1536	−0.2787	4.2593
β	0.2899	0.2959	−0.2922	0.872
λ	−0.2299	1.0181	−2.2324	1.7727
θ	2.5705	0.4928	1.6013	3.5397
σ	1.9237	1.7946	−1.6062	5.4535
a	2.5579	5.8507	−8.9498	14.0655
b	0.6719	0.5704	−0.4501	1.7938
δ	3.2272	5.8322	−8.2441	14.6984

B. Graphical Representations

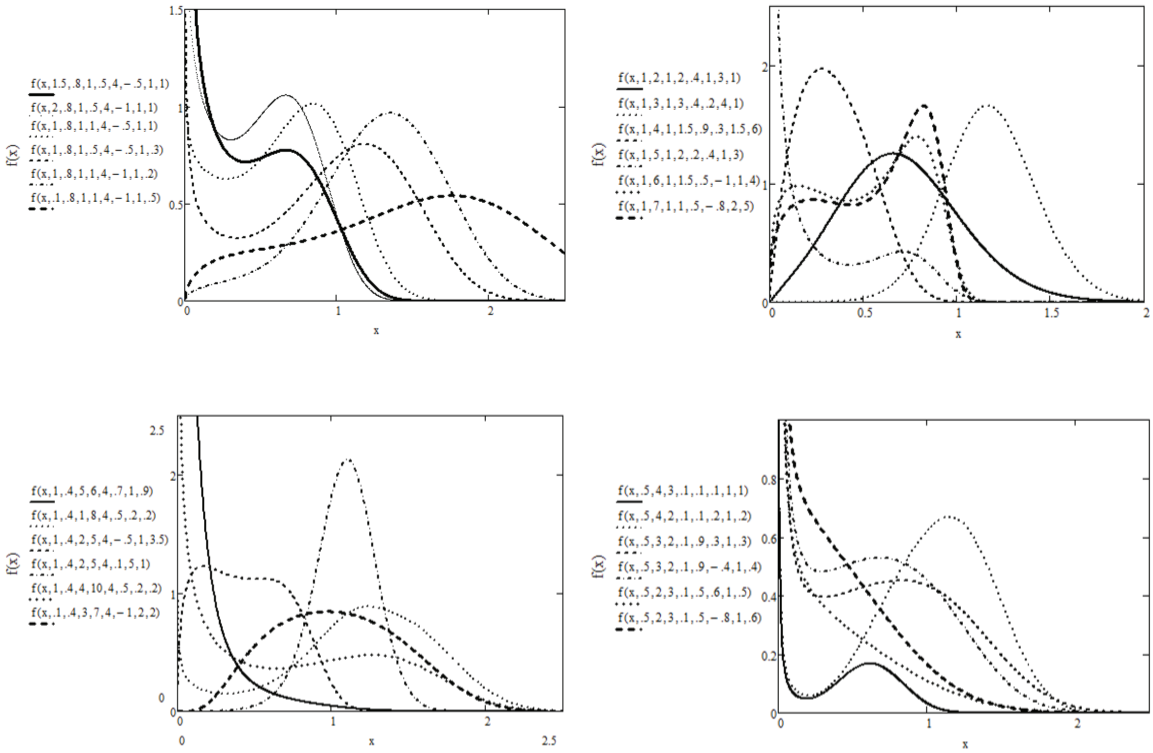

Figure 1. Density curves of Kw-TEAW model for different values of parameters.

Figure 2, lower and upper panels, provide some plotsof the Kw-TEAW hazard rate function, showing that it is quite flexible for modelling survival data.

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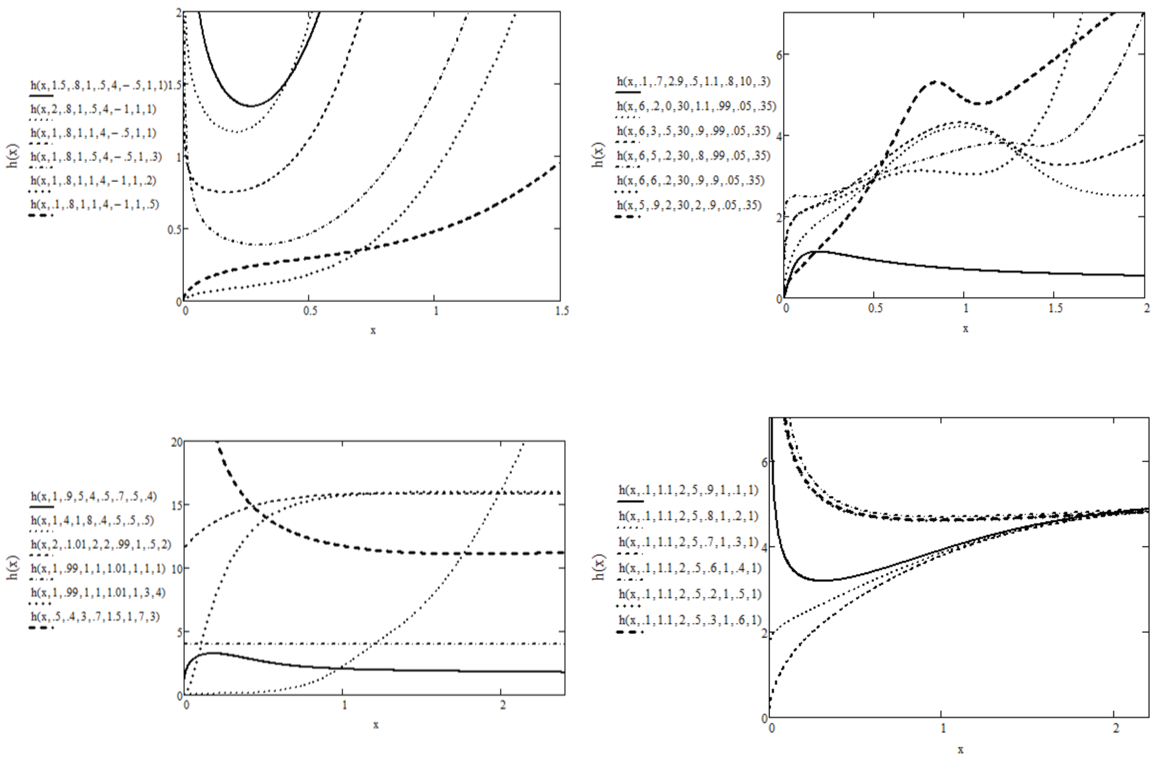

Figure 2. Hazard rate curves of Kw-TEAW model for different values of parameters.

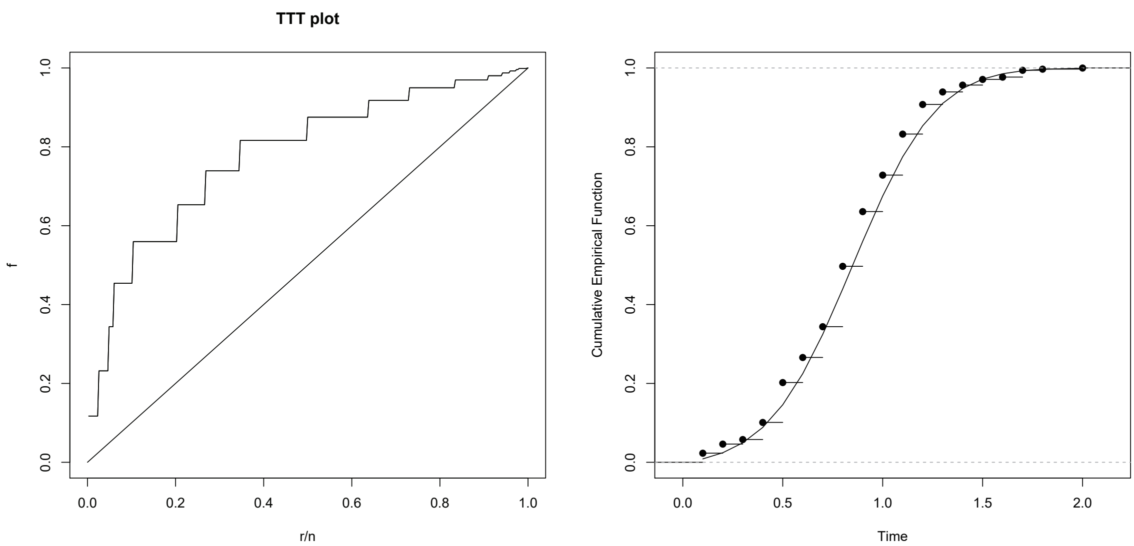

Figure 3. TTT-Plot and cumulative curves, empirical and estimated by Kw-TEAW model.

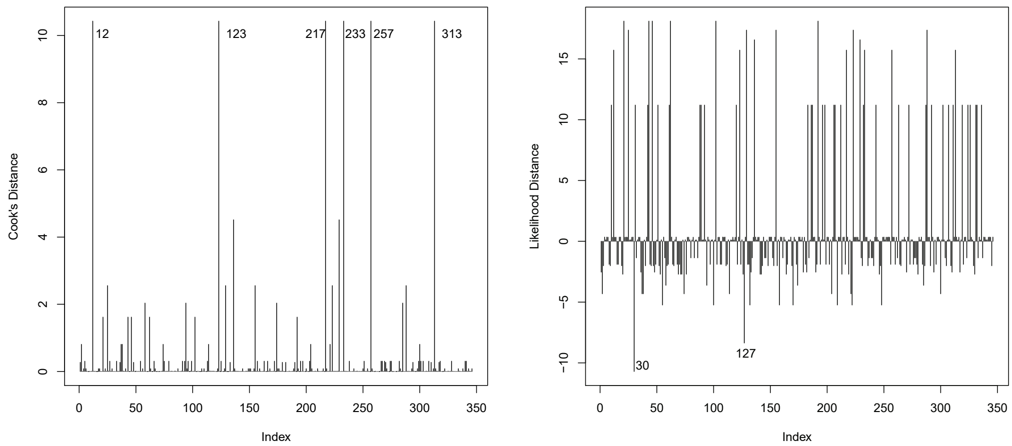

Figure 4. (a) Cooks's distance and (b) Likelihood distance.

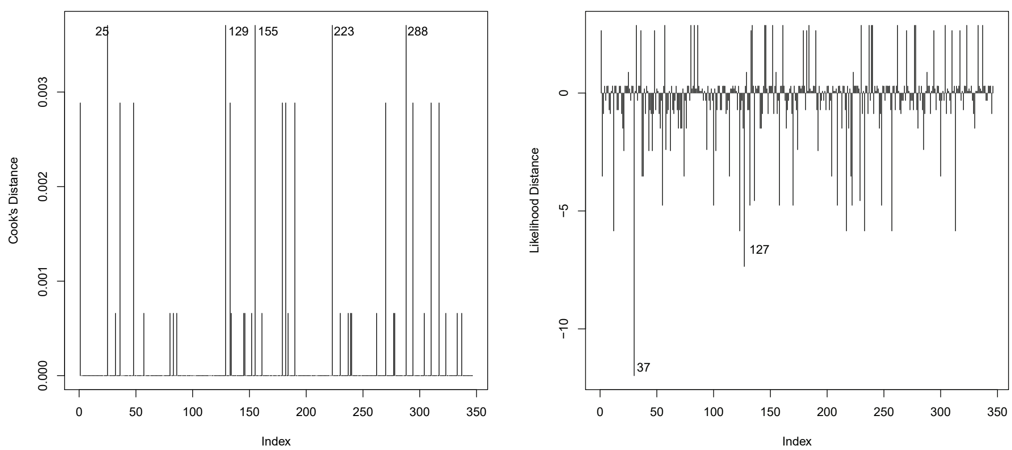

Figure 5. (a) Cooks's distance and (b) Likelihood distance after response perturbation.

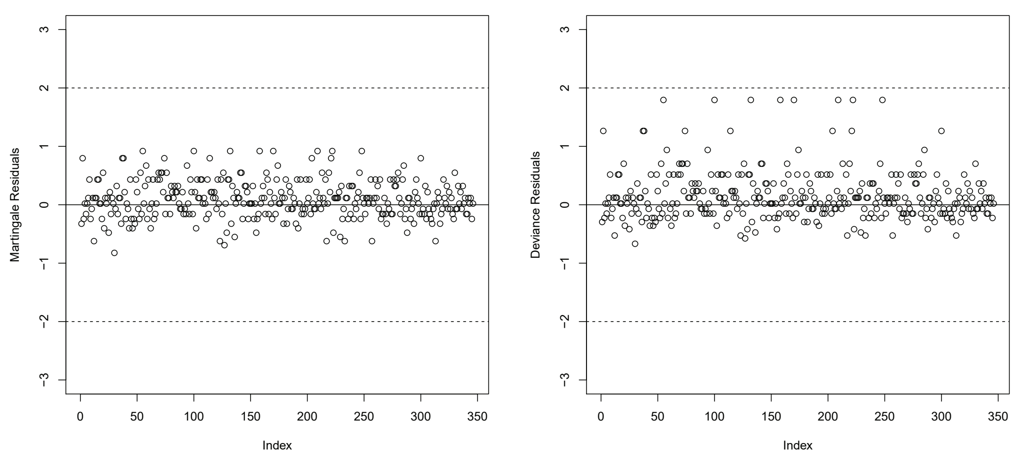

Figure 6. (a) Martingale residuals; (b) Deviance residuals.

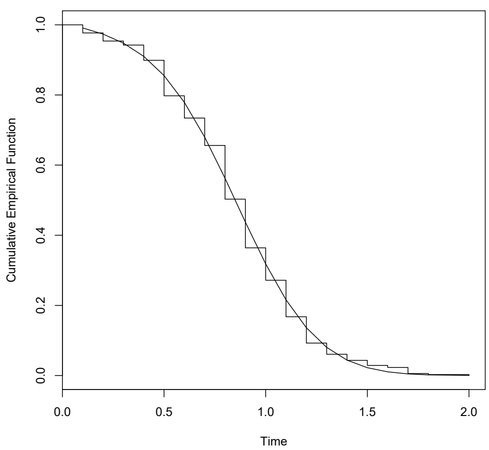

Figure 7. Kaplan-Meier empirical survival curve vs the re-adjusted model Kw-TEAW.